PERGAMON

International Journal of Heat and Mass Transfer 42 (1999) 3331–3336

International Journal of HEAT and MASS TRANSFER

Finite element method for simulation of 3-D form filling with incompressible fluid

I.H. Katzarov*

Bulgarian Academy of Science, Institute for Metal Science, 67 Shipchenski Prohod Str, 1574 Sofia, Bulgaria

Received 25 October 1998

Abstract

In the present article we propose an approach for treating the problem for the filling of a cavity with incompressible fluid by the Finite Element Method (FEM). The area occupied by fluid is approximated by elements, which have dimensions in both space and time. With this choice of elements the method is an implicit time stepping technique and adaptive to the non-stationary nature of the solution. An algorithm is developed for localization of the position of the free surface and for constructing of FE-mesh over the fluid occupied area at each time step. Numerical results were obtained for simulation of the process of filling of the cavity of the form used for production of 3-D axi-symmetric automobile wheels by the Counter Pressure Casting-method. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

The Finite Element Method (FEM) for solving the Navier–Stokes equation for problems with a free boundary is used in the velocity–pressure formulation. The elements chosen have dimensions in both space and time, and this choice of elements allows for the easy determination of the position of the free surface and also incorporates the natural boundary conditions in a straightforward manner [1–4]. The method is essentially an implicit time stepping technique and therefore, stable even for relatively large time steps.

Even at vanishing Reynolds number the convective term and the free surface generally make the equations non-linear and so a steady flow state requires an iterative computation that converges to that state. We use the Newton iteration process to solve the entire set of equations simultaneously for velocity and pressure [5,6].

Methods of this type have been used for slowly changing and time-independent free surface problems [4]. In this paper we propose an adaptation of the space–time FEM for treating the problem of filling a cavity with incompressible fluid. The numerical solution of the Navier–Stokes equations for this problem is complicated by the need to trace accurately, the path in time of the free surface and to generate FE mesh at the area occupied by the liquid at each time step. In Section 3 we propose a method for parameterisation and description of the evolution of the free surface and grid superposition–deformation procedure for updating of the FE mesh.

2. Mathematical formulation

In this section we shall give a short review of the method of space–time elements. The fluid equations

* Tel.: +359-2-7142-363; fax: +359-2-703-207.
E-mail address: ivo@ims.lines.acad.bg (I.H. Katzarov)

PII: S0017-9310(98)00380-9

to be solved are the momentum conservation equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \mathbf{f} + \nabla\cdot\sigma \tag{1}$$

where $\mathbf{u}$ is the velocity and $\mathbf{f}$ is the body force. For Newtonian flow the following equation applies

$$\sigma = -p\delta_{ij} + v\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

where p is the pressure and v is the kinematic viscosity of the fluid. For an incompressible fluid, the momentum Eq. (1), must be supplemented with the incompressibility condition

$$\nabla\cdot\mathbf{u} = 0. \tag{3}$$

If we use (2) and (3), then (1) may be rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot(\mathbf{uu} + p\mathbf{I}) - v\Delta\mathbf{u} = \mathbf{f}. \tag{4}$$

The weak form of the conservation equations may be written as

$$\int_V \phi\left(\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x_j}\left(u_j u_i + p\delta_{ij} - v\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{d}V = \int_V \phi f_i\,\mathrm{d}V \tag{5}$$

$$\int_{V(t)} \psi\frac{\partial u_j}{\partial x_j}\,\mathrm{d}v = 0 \tag{6}$$

where $V(t)$ is the region that the liquid occupies at time t and V the space–time domain for $t>0$

$$V = \{(x_i,t){:}x_i \in V(t), t > 0\}.$$

The functions ϕ and ψ are required to be measurable in Sobolev sense and ϕ vanishes on that part of the boundary of V which satisfies the no-slip boundary conditions.

If we use the Green's theorem, then (5) can be rewritten in the Galerkin form

$$\int_V \phi\left(\frac{\partial u_i}{\partial t} - \frac{\partial \phi}{\partial x_j}\left(u_j u_i + p\delta_{ij} - v\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{d}V + \int_{C(t)} \phi\left(u_j u_i + p\delta_{ij} - v\frac{\partial u_i}{\partial x_j}\right)n_j\,\mathrm{d}s\,\mathrm{d}t = \int_V \phi f_i\,\mathrm{d}V \tag{7}$$

where the integration in $\mathbf{s}$ is over the free boundary of $V(t)$ and $C(t)$ is the domain

$$C(t) = \{(x,t){:}x \in \delta V(t), t > 0\}.$$

The momentum equation requires conditions on every boundary. At solid boundaries there is no normal velocity and no appreciable tangential slip. The natural boundary conditions on the free surface is incorporated by the replacement of

$$\left(p\delta_{ij} - v\frac{\partial u_i}{\partial x_j}\right)$$

by

$$v\frac{\partial u_j}{\partial x_i}n_j - \gamma\left(\sum_{k=1}^{2}\frac{1}{R_k}\right)n_i$$

where R_k are the radii of the surface curvature of the interface in any two orthogonal planes containing the outward normal $\mathbf{n}$ with components n_k and γ is the coefficient of surface tension.

The set of conservation laws (6) and (7) form the system of equations to be solved using the FEM. Let us assume that the solution of the set is given at time t and the solution at some later time is desired. V^n is defined to be the region in the space–time containing the fluid between the times t^n and t^{n+1}

$$V^n = \{(x_i,t){:}x_i \in V(t), t^n \le t \le t^{n+1}\}.$$

Then (6) and (7) may be rewritten as

$$\int_{V(t^{n+1})} \psi\frac{\partial u_j}{\partial x_j}\,\mathrm{d}v = 0 \tag{8}$$

$$\int_{V^n} \phi\left(\frac{\partial u_i}{\partial t} - \frac{\partial \phi}{\partial x_j}\left(u_j u_i + p\delta_{ij} - v\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{d}V + \int_{C^n} \phi\left(u_j u_i + v\frac{\partial u_j}{\partial x_i} - \gamma\frac{1}{\mathbf{R}}\delta_{ij}\right)n_j\,\mathrm{d}s\,\mathrm{d}t = \int_{V^n} \phi f_i\,\mathrm{d}V \tag{9}$$

where

$$C^n = \{x_i, t){:}x_i \in \delta V(t), t^n \le t \le t^{n+1}\}.$$

We approximate the region $V(t^n)$ by a set of finite elements, in particular a set of eight-node elements in the 3-D case, and introduce parametric approximations, which map these elements onto a standard cube. For these transformations the nodes

$$P_i = (x_i, y_i, t_i) \quad i = 1, \ldots, 8$$

of each element are transformed into eight points with coordinates (1, 1, 1), (0, 1, 1), (0, 0, 1), . . . , (0, 0, 0) in the space (p, q, r). The standard transformation of the coordinates is written in the form

$$z = \sum_{i=1}^{8} \varphi_i(p,q,r)z_i \quad (z = x,y,t)$$

where

$$\varphi_1 = pqr, \quad \varphi_2 = (1-p)qr, \quad \varphi_3 = (1-p)(1-q)r,$$

$$\varphi_4 = p(1-q)r \quad \varphi_5 = pq(1-r),$$

$$\varphi_6 = (1-p)q(1-r), \quad \varphi_7 = (1-p)(1-q)(1-r),$$

$$\varphi_8 = p(1-q)(1-r).$$

Clearly the elements approximating the regions $V(t^n)$ and $V(t^{n+1})$ are the triangular bases of the prisms which approximate the region V^n. The subdivision of the domain V^n into a set of finite elements reduces the original problem to one, which is finite dimensional and the values of velocity, and pressure are calculated only at the nodes of the elements. In terms of basic function expansion the velocity and pressure fields are taken to be of the form

$$u_i \cong \tilde{u}_i = \sum_k \phi^k(x,y)u_k^{(i)}(t)$$

$$p \cong \tilde{p} = \sum_k \psi^k(x,y)p_k(t)$$

where

$$u_k^{(i)}(t) = (1-r)u_k^{(i)n} + ru_k^{(i)n+1}$$

$$p_k(t) = (1-r)p_k^n + rp_k^{n+1}$$

and $u_k^{(i)n}$, p_k^n are the nodal values of u and p at time t^n, ϕ^k and ψ^k are the basic functions which are assumed to form a complete set of functions over the fluid filled space. The interpolating functions ϕ^k must be chosen to preserve continuity of velocity between the elements because of the first-order derivatives in Eq. (9). In this paper ϕ^k are chosen to be a set of bilinear pyramid functions

$$\phi^k(P_l) = \delta_l^k = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}.$$

No continuity requirement is necessary for the interpolating functions ψ^k. They are chosen to be a set of step functions

$$\psi^k = \begin{cases} 1 & (x,y) \in E^k \\ 0 & (x,y) \notin E^k \end{cases}.$$

The Galerkin approximation satisfies

$$\int_{V(t^{n+1})} \psi^k \frac{\partial \tilde{u}_j}{\partial x_j}\, \mathrm{d}v = 0 \tag{10}$$

$$\int_{V^n} \phi^k \left(\frac{\partial \tilde{u}_i}{\partial t} - \frac{\partial \phi^k}{\partial x_j} \left(\tilde{u}_j \tilde{u}_i + \tilde{p}\delta_{ij} - \nu \frac{\partial \tilde{u}_i}{\partial x_j} \right) \right) \mathrm{d}V + \int_{C^n} \phi^k \left(\tilde{u}_j \tilde{u}_i + \nu \frac{\partial \tilde{u}_j}{\partial x_i} - \gamma \frac{1}{\mathbf{R}} \delta_{ij} \right) n_j \, \mathrm{d}s\, \mathrm{d}t = \int_{V^n} \phi^k f_i \, \mathrm{d}V. \tag{11}$$

Eqs. (10) and (11) form a system of equations

$$\mathbf{A} \cdot \mathbf{u}^{n+1} + \mathbf{B} \cdot u^{n+1} u^{n+1} + \mathbf{C}(\mathbf{u}^n) = \mathbf{0} \tag{12}$$

which can be solved for the unknown vector

$$\mathbf{u}^{n+1} = \begin{pmatrix} u_k^{(1)n+1} \\ u_k^{(2)n+1} \\ u_k^{(3)n+1} \\ p_k^{n+1} \end{pmatrix}, \quad k = 1, \ldots, N.$$

Note that even at vanishing Reynolds number the convective term $\mathbf{B}$ is non-linear in u. Eq. (12) is non-linear, so some iterative procedure is in general necessary for the solution of the above system. The approximation of $V(t^n)$ and $V(n^{n+1})$ is composed of the eight-node bases of the 16-node space–time elements for 3-D. For free boundary problems these elements will change in shape which in its turn depend on the values of velocity on the free surface at time t^{n+1}. The system of Eq. (12) and the method for computing the time evolution of the free surface are solved iteratively. The Newton–Raphson iterative procedure was adopted for this work.

3. Newton–Raphson method

The direct linearisation procedures for solving the non-linear equations proved to be unstable in a number of cases and thus, the Newton–Raphson procedure was adopted for this work. An error function f is defined as

$$f^m = \mathbf{A} \cdot \mathbf{u}_n^m + \mathbf{B} \cdot u_n^{(i)m} u_n^{(j)m} - \mathbf{C}$$

and the solution proceeds by solving the correction $\Delta\mathbf{u}_n$, to the previous estimate. Then

$$\mathbf{J}^m \Delta\mathbf{u}_n^{m+1} = f^m \tag{13}$$

where

$$J_{ij}^{m} = \frac{\partial f_{i}^{m}}{\partial u_j}$$

is the Jacobian of the system at step m and

$$\mathbf{u}_n^{m+1} = \mathbf{u}_n^{m} + \Delta \mathbf{u}_n^{m+1}.$$

The actual solution thus consists of evaluating all the error functions **f** and computing the corrections to **u** to eliminate all the errors.

The Newton–Raphson procedure has proved to be convergent for all problems that show acceptable answers in the first step. The first step in the fluid flow equations is essentially a linear problem where zero initial conditions are assumed.

In the present work Newton's method is applied to the entire weighted residual equation set, so that pressure and velocity are all found simultaneously. At element level the matrix problem (13) is

$$\begin{pmatrix} N^{(11)} & N^{(12)} & N^{(13)} & C^{(1)} \\ N^{(21)} & N^{(22)} & N^{(23)} & C^{(2)} \\ N^{(31)} & N^{(32)} & N^{(33)} & C^{(3)} \\ H^{(1)} & H^{(2)} & H^{(3)} & 0 \end{pmatrix} \cdot \begin{pmatrix} \Delta u^{(1)} \\ \Delta u^{(2)} \\ \Delta u^{(3)} \\ \Delta p \end{pmatrix} = \begin{pmatrix} f_{\mathrm{m}}^{(1)} \\ f_{\mathrm{m}}^{(2)} \\ f_{\mathrm{m}}^{(3)} \\ f_{\mathrm{c}} \end{pmatrix} \quad (14)$$

Here $\mathbf{f}_{\mathrm{m}}$ is the vector of residuals of the component momentum equations weighted with three linear basis functions; $\mathbf{f}_{\mathrm{c}}$ is that of residuals of the continuity equation weighted with step functions. The matrices of derivatives are

$$N^{(ij)} = \frac{\partial f_{\mathrm{m}}^{(i)}}{\partial u^{(j)}}, \quad C^{(i)} = \frac{\partial f_{\mathrm{m}}^{(i)}}{\partial p}, \quad H^{(i)} = \frac{\partial f_{\mathrm{c}}}{\partial u^{(i)}}.$$

Note that the zero occurs because the continuity equation is independent of the pressure.

The residual Eq. (14) is solved at each time step. The method is sufficiently accurate that at each time step only one Newton iteration is required for convergence.

4. The free boundary problem

The process of generating an FEM-mesh on the region occupied by the liquid, at each time step, involves three separate tasks. First, it is necessary to devise a method for describing the location and shape of the boundary. Second, an algorithm must be given for computing the time evolution of the boundary. Finally, deriving a method for updating the FE-mesh on the area occupied by the fluid in order to resemble the free surface more accurately.

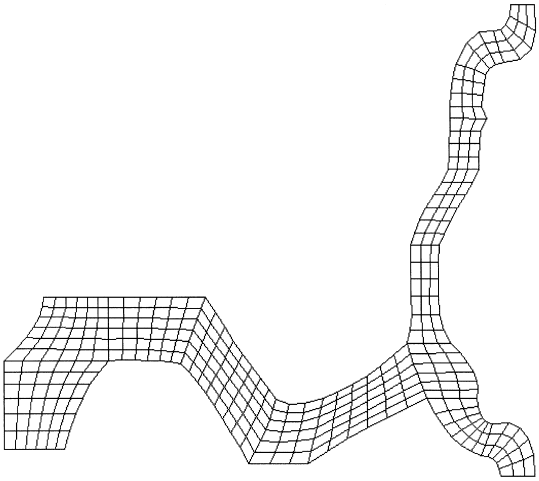

Fig. 1. Intitial FE grid on a form used for casting of automobile wheels.

The first two problems are related because the method of description will govern the choice of an evolution algorithm. To solve them we parameterise the free surfaces by chains of points connected by line segments. The set of points comprises the nodes of the

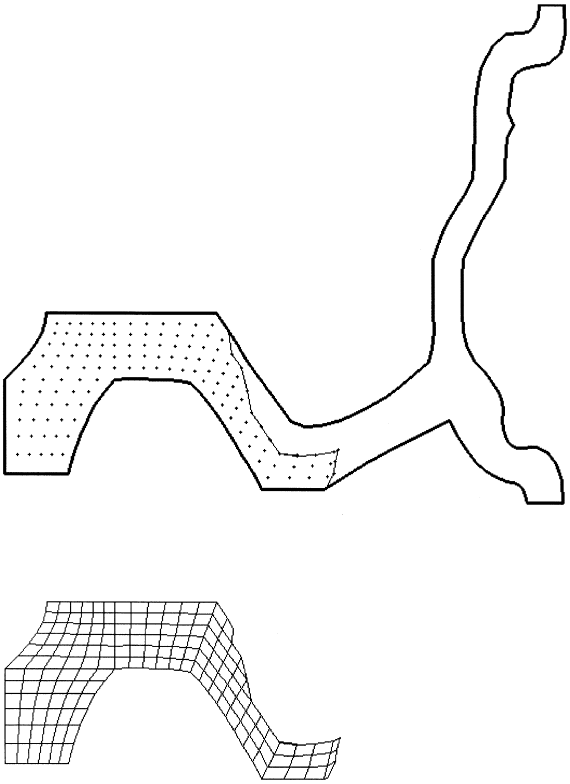

Fig. 2. FE grid constructed on the area occupied by fluid.

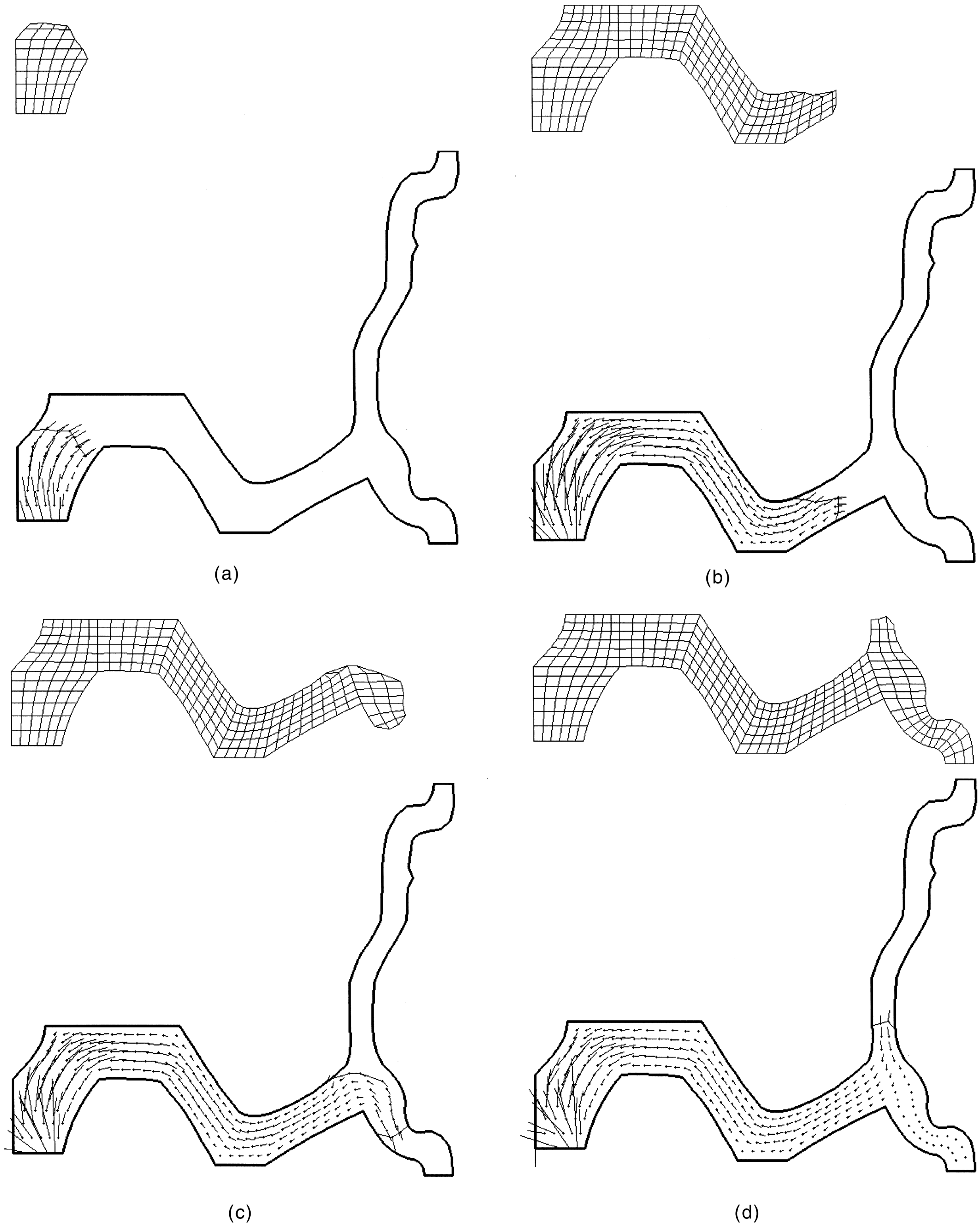

Fig. 3. Evolution of the velocity vectors and fluid configuration for automobile wheel casting problem.

FE mesh, situation on the free surface. The evolution of the chain of line segments is easily accomplished by simply moving each point with local fluid velocity.

After obtaining the new localisation of the free boundary we developed a grid superposition–deformation method in order to adjust the FE mesh adaptively and automatically to features of the flow. This method constructs a mesh on the $V(t)$ domain occupied by the liquid, essentially from the data of points on its contour and from an initial FE grid G, constructed on the whole cavity. At the first step the FE grid G is constructed on the cavity of filling in such a way as to contain $V(t)$ (Fig. 1). The process of generation of a new FE mesh on $V(t)$, which resembles the real geometry of the area occupied by fluid more accurately, is comprised of the following phases:

1. removal of boxes of G which do not intersect the domain $V(t)$;
2. a purely internal box becomes an element of the mesh;
3. processing of the elements of G containing a section of the boundary of the domain $V(t)$. These type of boxes (and the adjacent if it is necessary) have to be deformed in such a way that the points in which the free surface truncate G become nodes of the new grid. After modification these nodes are placed on the free surface. As a result, a new parameterisation of the free surfaces by chains of points connected by line segments is obtained (Fig. 2).

The procedure described above is repeated at each time step.

5. Numerical results

Numerical results have been obtained for simulation of the process of filling with molten metal, the cavity of the form used for casting 3-D axi-symmetric automobile wheels by the Counter Pressure Casting-method. The filling of the form is inflicted by the pressure difference between the chambers of a CPC unit (pressure is applied at the inlet and atmospheric pressure—on the boundary). The evolution of the free surface, the velocity field and the automatically generated FE mesh used for calculation of the velocity and pressure are shown in Fig. 3a–d. The velocity vectors are drawn from the nodes of the elements, which are marked by + signs.

References

[1] R. Bonnerot, P. Jamet, Int. J. Num. Meth. Eng. 8 (1974) 811–820.
[2] P. Jamet, R. Bonnerot, J. Comput. Phys. 18 (1975) 297–308.
[3] R. Bonnerot, P. Jamet, J. Comput. Phys. 25 (1977) 163–181.
[4] C.S. Frederiksen, A.M. Watts, J. Comput. Phys. 39 (1981) 282–304.
[5] I.P. King, W.P. Norton, K.R. Icemah, Finite Elements in Fluids, International Conference, University College of Wales, 1974, vol. 1.
[6] L.E. Scriven, S.F. Kistler, Coating flow theory by FE and asymptotic analysis of the Navier–Stokes systems, FE Flow Analysis, University of Tokyo Press, North-Holland, Amsterdam, 1982.